LSGNT Lunch Seminar

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UCL Geometry Seminar

Topics in differential geometry/geometric analysis

Gibbons-Hawking Ansatz

§1. The Hopf projection

$S^1 \curvearrowright \mathbb{R}^4 \simeq \mathbb{H}$ $\quad e^{i\theta}\cdot q = q e^{-i\theta}$ $\qquad \pi : \mathbb{H} \longrightarrow \operatorname{Im}\mathbb{H} \simeq \mathbb{R}^3$

free on $\mathbb{H}\setminus\{0\}$ $\qquad q \longmapsto q i \bar{q}$

$$q = z_1 + z_2 j \qquad \pi(z) = \left(|z_1|^2 - |z_2|^2,\ 2 z_1 z_2\right) \in \mathbb{R}\times\mathbb{C} \simeq \operatorname{Im}\mathbb{H}$$

$$\underline{x} \in \mathbb{R}^3$$

$$g_{\mathbb{R}^4} = \frac{1}{2|\underline{x}|}\left(dx_1^2 + dx_2^2 + dx_3^2\right) + 2|\underline{x}|\,\theta^2$$

$$h = \frac{1}{2|\underline{x}|} : \mathbb{R}^3\setminus\{0\} \longrightarrow \mathbb{R}_+ \qquad \theta \leftrightsquigarrow \mathcal{H} = \xi^{\perp}$$

v. field generating S^1-action

$$*_{\mathbb{R}^3} dh = \sum_{i=1}^{3} \frac{\partial h}{\partial x_i}\, dx_j \wedge dx_k = -d\theta$$

(ijk) cyclic (123)

§2. Hyperkähler 4-manifolds

$(M^4, \omega_1, \omega_2, \omega_3)$ $\qquad \bullet\ d\omega_i = 0$

$\omega_i \in \Omega^2(M)$ $\qquad \bullet\ \omega_i \wedge \omega_j = \delta_{ij}\, \omega_1^2$

$$T_x^* M \simeq \mathbb{R}^4 = \langle e_0, e_1, e_2, e_3 \rangle \qquad \omega_i = e_0 \wedge e_i + e_j \wedge e_k$$

Choose a direction in $\mathbb{R}^3$, say $(1,0,0)$

$\omega = \omega_1 \qquad \omega_c = \omega_2 + i\omega_3$

- $\omega_c \wedge \omega_c = 0 \rightsquigarrow$ almost cplx str. J

  loc. $\omega_c = \gamma_1 \wedge \gamma_2$

  $\Lambda^{1,0} = \langle \gamma_1, \gamma_2 \rangle \subset T^*M \otimes \mathbb{C}$

- $\omega \wedge \omega_c = 0 \rightsquigarrow \omega$ type $(1,1)$ wrt J

  $(\Rightarrow \omega \wedge \overline{\omega}_c = 0)$

$\rightsquigarrow$ Riemannian metric $g_{J,\omega}$

$d\omega_c = 0 \iff J$ integrable $+$ ω_c holomorphic symplectic form

$d\omega = 0 \iff (g, J, \omega)$ Kähler

$\omega^2 = \frac{1}{2}\omega_c \wedge \overline{\omega}_c \iff \mathrm{Ric}(g) = 0$

§3. The Gibbons-Hawking Ansatz

$U \subseteq \mathbb{R}^3$ open $\qquad M \to U$ principal S^1-bundle w/ connection θ

$h : U \to \mathbb{R}_+$

$$\rightsquigarrow \quad g = h\,(dx_1^2 + dx_2^2 + dx_3^2) + h^{-1}\theta^2$$

$$\omega_i = \theta \wedge dx_i + h\, dx_j \wedge dx_k \qquad \omega_i \wedge \omega_j = \delta_{ij}\,\omega_1^2$$

$$d\omega_i = 0 \iff *_{\mathbb{R}^3} dh = -d\theta$$

Note: everything determined by h

$d * dh = d(-d\theta) = 0 \Rightarrow h$ positive harmonic fct on $U \subseteq \mathbb{R}^3$

$\frac{\partial^2 h}{\partial x_1^2} + \frac{\partial^2 h}{\partial x_2^2} + \frac{\partial^2 h}{\partial x_3^2} = 0$

$\partial x_i \quad \partial x_j \quad \partial x_3^2$

$$[*dh] \in 2\pi \operatorname{im}\left(H^2(U;\mathbb{Z}) \to H^2(X;\mathbb{R})\right)$$

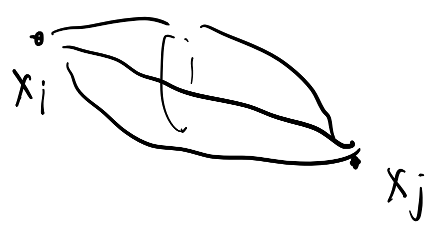
Examples

- $U = \mathbb{R}^3 \quad h = m \in \mathbb{R}_{>0} \quad \rightsquigarrow M = \mathbb{R}^3 \times S^1_{2\pi/\sqrt{m}}$
- $U = \mathbb{R}^3 \setminus \{0\} \quad h = \frac{1}{2|\underline{x}|} \quad \rightsquigarrow M = \mathbb{R}^4$
- $U = \mathbb{R}^3 \setminus \{0\} \quad h = m + \frac{1}{2|\underline{x}|} \quad \rightsquigarrow M = \mathbb{R}^4$ Taub-NUT
- $U = \mathbb{R}^3 \setminus \{x_1, \dots, x_{k+1}\} \quad h = m + \sum_{i=1}^{k} \frac{1}{2|x - x_i|}$

(-2) curves

$\rightsquigarrow M \underset{\text{diffeo}}{\simeq}$ minimal solution of $\mathbb{C}^2/\mathbb{Z}_k$

$\{xy = z^{k-1}\} \subset \mathbb{C}^3$



$\mathbb{C}^2/\Gamma \quad \Gamma < SU(2)$ finite

$\Gamma =$ ADE classification

$A_k \leftrightsquigarrow \Gamma = \mathbb{Z}_{k+1}$

$D_k \rightsquigarrow \Gamma$

E_6, E_7, E_8